P. Pastini and C. Zannoni: TABLES OF GLEBSCH-GORDAN COEFFICIENTS FOR INTEGER ANGULAR MOMENTUM
\( J = 0 \div 6 \)
TABLES OF CLEBSCH-GORDAN COEFFICIENTS FOR INTEGER ANGULAR MOMENTUM $J \leq 4.6$

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1. INTRODUCTION

Clebsch-Gordan coefficients play an essential role in a variety of problems involving addition of angular momenta and in general tensor manipulation (7-12). Apart from more conventional applications in quantum mechanics Clebsch-Gordan coefficients are now often employed in material science (13, 14) and in the statistical mechanics of condensed phases and in particular of anisotropic fluids such as liquid crystals (15). All of these practical applications require knowledge of the explicit values of Clebsch-Gordan coefficients of integer rank. To this effect there are already, of course, both general formulas and tables either of Clebsch-Gordan coefficients or of the closely related $3j$ coefficients (20-22) as well as computer programs for their evaluation (23, 24). In principle it is therefore not too difficult to obtain a certain required set of Clebsch-Gordan coefficients. In practice, however, this may still prove rather laborious. In particular the use of most tables, even if available for the ranks of interest, may still result in a somewhat time consuming and error prone exercise. For example the most commonly available tables only go up to an integral angular momentum of two (22) or four (20) and they either give the coefficients in floating point form or in terms of a...
string of exponents of prime numbers whose product gives the desired coefficient. Moreover, they tend to make full use of the various symmetries of which the Clebsh-Gordan symbols are endowed\(^3\). While this can be advantageous for compactness, we think that the aim of a set of tables should be that of giving coefficients quickly while limiting the chances of trivial mistakes so that some redundancy is advisable. In the present set of tables, we have chosen therefore to list Clebsh-Gordan coefficients in the most straightforward and immediately usable form, trading some extension in size against convenience of the user. In the next section, the notation employed here is defined and contact is made with other widespread conventions. Some often used formulas of angular momentum and irreducible tensors (Tensort) are also listed for completeness and easy reference. Coefficients for integer angular momentum of rank up to six are listed, since there is how a number of applications in molecular physics, ranging from calculation of higher terms in intermolecular potentials\(^{25-27}\) to evaluation of matrix elements arising in multiphoton spectroscopy\(^{14}\), where these are required. We quote as an example the theory of hyper-Raman effect\(^{128}\), where rotation al averages of sixth rank tensors are involved. We shall give Clebsh-Gordan coefficients both in exact form, i.e., as square roots of fractional numbers and in floating point form. We are not aware of other tables listing Clebsh-Gordan coefficients up to this rank in this form.

2. - NOTATIONS

There is an impressive number of different conventions\(^{28-41}\) for writing Clebsh-Gordan coefficients, even though many of them only differ for the symbols employed. We choose here to define a Clebsh-Gordan coefficient according to the phase convention of Rose\(^{11}\). I.e., we write the coupling coefficient between two states of angular momentum \(J_1\) and \(J_2\) to yield a state \(|J_3 m_3\rangle\) as

\[
|J_3 m_3\rangle = \sum_{m_1 m_2} C(J_1, J_2, J_3; m_1, m_2, m_3) |J_1 m_1\rangle |J_2 m_2\rangle
\]  

where \(C(J_1, J_2, J_3; m_1, m_2, m_3)\) is a Clebsh-Gordan vector coupling coefficient and \(J_1, J_2, J_3\) can take non-negative integer or semi-integer values.

Clebsh-Gordan coefficients are real and can be generally written in terms of square roots of ratios of integers. In the Tables given in section 4, reproduced from computer printouts, we employ the notation

\[
C(J_1, J_2, J_3; m_1, m_2, m_3) \propto \mathbb{R}(N1/N2) \propto \text{sgn}(N1)\left\{\frac{(N1)/(N2)}{1/2},
\right.
\]

\]
where \( J_1, J_2, J_3 \) and \( N_1, N_2 \) are integers and the function \( \text{sgn}(M) \) gives the sign of its argument \( M \). As mentioned before, the results are also given for convenience in floating point form rounded to eight decimal places. The angular momentum values \( J_1, J_2, J_3 \) are said to form a triangle \( \mathcal{A}(J_1, J_2, J_3) \), in the sense that the following relations hold for the allowed values

\[
\begin{align*}
\mathcal{A}(J_1, J_2, J_3) \colon & \quad J_1 + J_2 + J_3 \geq 0 \quad (3a) \\
& \quad J_1 - J_2 + J_3 \geq 0 \quad (3b) \\
& \quad -J_1 + J_2 + J_3 \geq 0 \quad (3c)
\end{align*}
\]

where \((J_1 + J_2 + J_3)\) is an integer. The triangular relation is symmetric in the three angular momenta. Clebsch-Gordan coefficients formed with combinations of angular momenta not satisfying this rule are equal to zero and, of course, are not reported in the tables. The angular momentum projection values \( m_1, m_2, m_3 \) can take the values

\[
m_1 = -J_1, -J_1 + 1, \ldots, J_1; \quad m_2 = -J_2, -J_2 + 1, \ldots, J_2; \quad m_3 = -J_3, -J_3 + 1, \ldots, J_3. \quad (4)
\]

Other common notations for the same coefficients are listed below (cf. Ref. (2) and (11)).

\[
\begin{align*}
& J_1 J_2 J_3 \quad C_{J_1 J_2 J_3}^{m_1 m_2 m_3} \\
& J_1 J_2 J_3 \quad C_{m_1 m_2 m_3}^{J_1 J_2 J_3}
\end{align*}
\]

Alder(32), Jahn(33a), Jahn and Hope(33b)

Biedenharn(35), Redondo(36), Simon(39)

Blatt and Welskopf(8)

Boys and Sahn(41)

Brink and Satchler(5), Maseger(40), Roybarski(41), Jerphagnon(14)

Condon and Shortley(6)

Eckart(37)
Explicit relations for the calculation of Clebsch-Gordan coefficients have been derived by Wigner\(^{(1)}\):

\[
\begin{align*}
C(J_1, J_2, J_3; m_1, m_2, m_3) &= \delta_{m_2, m_1 + m_2} \\
&\times \left\{ \frac{(J_3 + 1)}{2 \Gamma(J_3 + 1)} \right\}^{1/2} \\
&\times \sum_{v} \frac{(-1)^{J_3 + J_2 + m_2}}{\sqrt{v(J_3 + J_2 + m_3 + 1)(J_3 + J_2 + m_3 + v)}} \\
&\times \left\{ (J_3 + J_2 + m_3 + 1) \right\}^{-1/2}
\end{align*}
\]  

and by Racah\(^{(16)}\):

\[
\begin{align*}
C(J_1, J_2, J_3; m_1, m_2, m_3) &= \delta_{m_2, m_1 + m_2} \\
&\times \left\{ \frac{(J_3 + 1)}{2 \Gamma(J_3 + 1)} \right\}^{1/2} \\
&\times \sum_{v} \frac{(-1)^{J_3 + J_2 + m_2}}{\sqrt{v(J_3 + J_2 + m_3 + 1)(J_3 + J_2 + m_3 + v)}} \\
&\times \left\{ (J_3 + J_2 + m_3 + 1) \right\}^{-1/2}
\end{align*}
\]
In eqs. (5), (6) the index \( v \) takes all the integral values leaving the argument of the various factorial \( v \) non-negative.

Clebsch-Gordan coefficients are related to the often used and more symmetric 3j symbols introduced by Wigner\(^7\):

\[
C(J_1, J_2, J_3; m_1, m_2, m_3) = (-1)^{J_1+J_2-J_3} \binom{J_1 J_2 J_3}{m_1 m_2 m_3} \sqrt{(2J_1+1)(2J_2+1)}
\]

(7)

3. SOME USEFUL RELATIONS

We report here (Sections 3.1-3.4) for easy reference some useful properties of Clebsch-Gordan coefficients and (Section 3.5) a small collection of frequently employed formulas involving vector coupling coefficients. Applications to Wigner matrices and irreducible tensors are given in Sections 3.7 and 3.8.

3.1. Symmetries

There are various symmetry relations that can be derived e.g. from the general explicit expression for the Clebsch-Gordan coefficients given by Racah\(^1,16\). We have in particular:

\[
C(J_1, J_2, J_3; m_1, m_2, m_3) =
\]

\[
\begin{align*}
&= (-1)^{J_1+J_2-J_3} C(J_1, J_2, J_3; -m_1, -m_2, -m_3) \\
&= (-1)^{J_1+J_2-J_3} C(J_2, J_1, J_3; m_2, m_1, m_3) \\
&= (-1)^{J_1+1} \sqrt{(2J_1+1)(2J_2+1)} C(J_1, J_2, J_3; m_1, m_2, m_3)
\end{align*}
\]

(8a) (8b) (8c)

From these relations some other useful equations can in turn be derived

\[
C(J_1, J_2, J_3; m_1, m_2, m_3) =
\]

\[
\begin{align*}
&= (-1)^{J_1+J_2} \sqrt{(2J_1+1)(2J_2+1)} C(J_2, J_3, J_1; m_2, m_3, m_1) \\
&= (-1)^{J_1+J_2} \sqrt{(2J_1+1)(2J_2+1)} C(J_3, J_2, J_1; m_3, m_2, m_1) \\
&= (-1)^{J_1+J_2} \sqrt{(2J_1+1)(2J_2+1)} C(J_3, J_2, J_1; m_3, m_2, m_1)
\end{align*}
\]

(9a) (9b) (9c)
3. 2. Orthogonality

The Clebsch-Gordan coefficients are elements of a unitary transformation and they satisfy orthogonality relations. These can be written as

\[ \sum_{m_1, m_2} C(j_1, j_2, j; m_1, m_2, m_1, m_2, m, m) = \delta_{j_1j_2} \delta_{m_1m_2} \delta_{m_1m} \delta_{m_2m} \]  

or

\[ \sum_{m_1, m_2} C(j_1, j_2, j; m_1, m_2, m_1, m_2, m_1, m_2, m, m) = \delta_{j_1j_2} \delta_{m_1m_1} \delta_{m_2m_2} \delta_{m_1m} \delta_{m_2m} \]

We also have

\[ \sum_{m_1, m_2} C(j_1, j_2, j; m_1, m_2, m_1, m_2, m_1, m_2, m, m) = \delta_{m_1m_1} \delta_{m_2m_2} \delta_{m_1m} \delta_{m_2m} \]

or

\[ \sum_{m_1, m_2} C(j_1, j_2, j; m_1, m_2, m_1, m_2, m_1, m_2, m, m) = \delta_{m_1m_1} \delta_{m_2m_2} \delta_{m_1m} \delta_{m_2m} \]

3. 3. Sum rules

Some useful formulas are:

\[ \sum_{m} C(j_1, j_2, j; m, 0, -m) C(j_1, j_2, j; m, -M, 0, M) = \frac{(-1)^{2j_1+2j_2-2j+2j_1j_2j}}{(2j+1)!} \left[ \frac{(2j_1+1)(2j_1+1)!}{(2j_2+1)!} \right]^{1/2} \]  

(cf. Ref. 42)

\[ m, \quad m = C(j_1, j_2, j; m_1, m_2, m)^2 = (2j+1) \]  

\[ \sum_{m} (-1)^m C(j_1, j_2, j; m, 0, -m, 0) = (-1)^{j_1} (2j+1)^{1/2} \delta_{j,j_1} \]  

Sethborn and Filler(44) have derived:

\[ \sum_{j_1} \left[ C(j_1, j_2, j_1; 0, 0, 0) \right]^2 - (2j_1+1) \left[ (j_1, j_2, j_1; 0, 0, 0) \right]^2, \left[ (j_1, j_2, j_1; 0, 0, 0) \right]^2 \]  

where \((-1)! \times 1) is implied. A few recent results are: Din's formula(44, 45)

\[ \sum_{j_1} (2j_1+1) \left[ (j_1, j_2, j_1; 0, 0, 0) \right]^2, \left[ (j_1, j_2, j_1; 0, 0, 0) \right]^2 = 0; \]

\[ J_1 \neq J_2, J_3 \]
where \( J_3, J_2 \in k \mathbb{Z} \) and \( k \neq 2, -2, 0 \), odd, and the following two obtained by Morgan [46]:

\[
\begin{align*}
J_1 \left\{ \begin{array}{l}
\sum_{J_2 = 0}^{J_1} (-1)^{J_2} C(J_1, J_2; J_1, J_2; m_0, m_0, J_1) \frac{J_2}{(2J_2 + 1)(2J_1 + 1)} \times \left\{ \frac{2J_1 + 1}{2J_1 + 1} \right\} \times \left\{ \frac{2J_2 + 1}{2J_2 + 1} \right\} \\
\end{array} \right.
\end{align*}
\]

(17)

\[
\begin{align*}
J_1 \left\{ \begin{array}{l}
\sum_{J_2 = 0}^{J_1} (-1)^{J_2} C(J_1, J_2; J_1, J_2; m_0, m_0, J_1) \frac{J_2}{(2J_2 + 1)(2J_1 + 1)} \times \left\{ \frac{2J_1 + 1}{2J_1 + 1} \right\} \times \left\{ \frac{2J_2 + 1}{2J_2 + 1} \right\} \\
\end{array} \right.
\end{align*}
\]

(18)

\[
\begin{align*}
= \left\{ \begin{array}{l}
1; \text{ if } J_1 = 0 \\
\left\{ \frac{2J_1 + 1}{2J_1 + 1} \right\} \times \left\{ \frac{2J_2 + 1}{2J_2 + 1} \right\}; \text{ if } J_1 \text{ is a positive integer number.}
\end{array} \right.
\end{align*}
\]

3.4 - Recurrence relations

We give here two recurrent equations [11] that may prove useful in further extending the present Tables if necessary. The first allows changing the angular momentum \( J \)

\[
\begin{align*}
\sum_{J_2 = 0}^{J_1} (-1)^{J_2} C(J_1, J_2; J_1, J_2; m_1, m_1, J_2) \times \frac{J_2}{(2J_2 + 1)(2J_1 + 1)} \times \left\{ \frac{2J_1 + 1}{2J_1 + 1} \right\} \times \left\{ \frac{2J_2 + 1}{2J_2 + 1} \right\} \\
= \left\{ \begin{array}{l}
\left\{ \frac{2J_1 + 1}{2J_1 + 1} \right\} \times \left\{ \frac{2J_2 + 1}{2J_2 + 1} \right\}; \text{ if } J_1 \text{ is a positive integer number.}
\end{array} \right.
\end{align*}
\]

(19)

The second relates Clebsch-Gordan coefficients with the same angular momentum \( J_1, J_2, J \) but different components:

\[
\begin{align*}
\sum_{J_2 = 0}^{J_1} (-1)^{J_2} C(J_1, J_2; J_1, J_2; m_1, m_1, J_2) \times \frac{J_2}{(2J_2 + 1)(2J_1 + 1)} \times \left\{ \frac{2J_1 + 1}{2J_1 + 1} \right\} \times \left\{ \frac{2J_2 + 1}{2J_2 + 1} \right\} \\
= \left\{ \begin{array}{l}
\left\{ \frac{2J_1 + 1}{2J_1 + 1} \right\} \times \left\{ \frac{2J_2 + 1}{2J_2 + 1} \right\}; \text{ if } J_1 \text{ is a positive integer number.}
\end{array} \right.
\end{align*}
\]

(20)

Recurrent relations especially useful for large \( J \) (\( \approx 30 \)) angular momentum have been obtained by Schulten and Gordon [19] both for 3j and 6j symbols.
5.5 - Some special formulae

Formulas giving certain classes of vector coupling coefficients in algebraic form can be obtained specializing the general eqs. (5) and (6). Explicit formulas for coefficients with one of the angular momentum rank $J+1$ can be found in the celebrated book by Condon and Shortley (6). As for semi-integer ranks, formulas for $J=1/2$ are reported, e.g. by Rose (11); while formulas for $J=3/2$, 5/2 are given by Saito and Mori (47). Here we present a small collection of relations mostly chosen according to what we have found most useful.

\[
C(J, J', 0, m, -m, 0) = (-1)^{m} \frac{\delta_{Jm}}{(2J+1)^{1/2}}
\]  
(21)

\[
C(J, J', m_{1}, m_{2}, m_{1}, m_{2}) = \frac{\delta_{J_{1}J_{2}}}{2J_{1}+1} \frac{\delta_{m_{1}m_{2}}}{2J_{2}+1}
\]  
(22)

\[
C(1,1,0, m, -m, 0) = (-1)^{m} m^{1/2}
\]  
(23)

\[
C(1,1,1, m, -m, 0) = m^{1/2}
\]  
(24)

\[
C(1,1,2, m, -m, 0) = (1/2) m^{1/2}
\]  
(25)

\[
C(J, J, J, 0, m, m) = C(J, J, J, 0, m, 0, m) = ( -m )^{1/2} \quad \text{if } J > 0
\]  
(26)

\[
C(3,0, 0, m, m) = C(3, 3, 1, m, m) = \left( \frac{3J+1}{(2J+1)} \right) \frac{1}{m} \quad \text{if } m > 0
\]  
(27)

\[
C(3, 1, J, 1, m, 0) = C(1, J, 3, 1, m, 0) = \left( \frac{(J+1)(2J+1)}{2J+1} \right)^{1/2} \quad \text{if } J > 0, m > 0
\]  
(28)

\[
C(3, 2, 0, m, m) = ( -m )^{1/2} \quad \text{if } J = 2, 4, 6 \quad \text{and zero otherwise}
\]  
(29)

\[
C(4,4,2, m, m) = ( -m )^{1/2} C(4, 2, 4, m, 0, m) = ( -m )^{10} \frac{(2m+2)}{(6J+1)^{1/2}}
\]  
(30)

\[
C(J, J, 3, m, 0, m) = \left( \frac{(J+1)(5J+6)}{2J+1} \right)^{1/2}
\]  
(31)

\[
C(J, J, 3, m, 0, 0) = ( -m )^{1/2} \frac{(J+1)(5J+4)}{(2J+1)^{1/2}}
\]  
(32)

\[
C(4, 2, 0, m, m) = ( -m )^{1/2} \frac{(5J+3)(5J+m+2)(2J+2)(5J+2)(2J+1)}{(2J+1)(2J+3)(2J+1)(2J+5)(2J+7)}
\]  
(33)

\[
C(J, J, 3, m, 0, m) = \left( \frac{(J+1)(5J+5)^{1/2}}{(2J+1)^{1/2}} \right)^{1/2}
\]  
(34)
\[
C(J, \lambda_1, m_1, m_2, m_3, J_3) = \left\{ \frac{(J + m_1 + 6)(J + m_2 + 6)(J + m_3 + 6)(J + m_1 + 4)(J + m_2 + 4)(J + m_3 + 4)(J + m_1 + 2)(J + m_2 + 2)(J + m_3 + 2)}{(2J + 1)(2J_1 + 1)(2J_2 + 1)(2J_3 + 1)(2J_1 + 3)(2J_2 + 3)(2J_3 + 3)} \right\}^{1/2} \quad (35)
\]

Eqn. (34), (35) have been given, albeit incorrectly, in Ref. (48).

\[
C(J, \lambda_1, \lambda_2, m_1, m_2, m_3, m_4) = \frac{(J_1 + m_1 + 6)(J_2 + m_2 + 6)(J_1 + m_1 + 4)(J_2 + m_2 + 4)(J_1 + m_1 + 2)(J_2 + m_2 + 2)}{(2J_1 + 1)(2J_2 + 1)(2J_1 + 3)(2J_2 + 3)(2J_1 + 5)(2J_2 + 5)} \quad (36)
\]

\[
C(J, \lambda_1, \lambda_2, \lambda_3, m_1, m_2, m_3, m_4) = \frac{1}{(2J_1 + 1)(2J_2 + 1)(2J_3 + 1)} \quad (37)
\]

where \( I(x) = (\pi/2) \left( x^2 / \pi \right)^{1/2} \), if \( J_1^2 + J_2^2 + J_3 \) is an even integer.

3.6. Asymptotic results

A classical result due to Hoggard and Toloehoff (49):

\[
C(J, \lambda_2, J; m_1, m_2, m_3) \propto \gamma^{J_2 + 1} \gamma^{J_1 + J_3} \frac{1}{m_1 J_3} \quad (38)
\]

where the small Wigner matrix \( C^{m_1}_{m_2} \) is defined in (1) and \( \cos \theta = m/J; \ J \gg 1, \ J_1 \ll J \) and of course \( m = m_1 + m_2 \).

3.7. The coupling of Wigner rotation matrices

Wigner rotation matrices or generalized spherical harmonics \( D^J_{m_1 m_2} (\alpha \beta \gamma) \) represent matrix elements of the operator performing a coordinate system rotation of Euler angles \( (\alpha \beta \gamma) \) in an angular momentum basis. Thus following Rose (11) convention

\[
D^J_{m_1 m_2} (\alpha \beta \gamma) = \langle Jm | \exp(-im_1) \exp(-ip_2) \exp(-ip_3) | Jn \rangle \quad (39)
\]

where \( 0 \leq m \leq 2J \), \( 0 \leq p \leq 2 \pi \). The Wigner rotation matrices form an orthogonal basis set in the Euler angles space, as such they are often used for writing down expansions of anisotropic quantities in the molecular theories of crystals (14), liquid crystals (15) and polymers (13).

Clebsch-Gordan coefficients arise naturally when we want to rewrite a product of Wigner rotation matrices (1) of the same argument and of rank \( J_1, J_2 \) in terms of a single rotation matrix. The coupling rule for these matrices can be written as

\[
\frac{1}{(2J_1 + 1)(2J_2 + 1)(2J_1 + 3)(2J_2 + 3)(2J_1 + 5)(2J_2 + 5)} \quad (37)
\]
where the $\text{LAB}$ and $\text{MCL}$ subscripts refer to laboratory and rotated or "molecular" frame. The components $T^{J,m}$ of a rank $J$ irreducible tensor verify the Racah(18) relations:

\begin{align}
  T^{J,m}_m &= \sum_{m_1} C^{J,m,m_1}_{m,m_1} T^{J,m_1}_m, \\
  T^{J,m}_{m_1} &= \left[ (2m)(J+m) \right]^{1/2} T^{J,m_1}_{m_1},
\end{align}

where the $x$ superscript indicates the commutation superoperator $A^x \equiv [A, \mathbb{H}]$, while $J_2$, $J_3$ are the usual angular momentum projection operators. Eqs. (47), (48) can be written more concisely as:

\begin{equation}
  T^{J,m}_m = (-)^{J} C(J, J_2, J_3, m, m_1, m_2, m_3) T^{J,m_1}_{m_1}, \quad n = 0, \pm 1
\end{equation}

A tensor of rank $J$ can be constructed from two tensors of rank $J_2$ and $J_3$ when they are coupled as follows:

\begin{equation}
  T^{J,m}_m(A_2, A_3) = \sum_{m_1} C(J_2, J_3, J, m, m_1, m_2, m_3) T^{J,m_2}_{m_2}(A_2) T^{J,m_3}_{m_3}(A_3)
\end{equation}

where the symbols $A_2$ and $A_3$ represent all the variables upon which the tensors depend.

3.3. - Wigner-Eckart theorem

The calculation of matrix elements $<J_1,m_1|T^{J,m}|J_2,m_2>$ of an irreducible tensor operator $T^{J,m}$ over an angular momentum basis set is simplified by the Wigner-Eckart theorem(11) according to which:

\begin{equation}
  <J_1,m_1|T^{J,m}|J_2,m_2> = K_{J_1J_2} C(J_2, J_3, J, m_2, m_3; m_2, m_3, m_1)
\end{equation}

where the quantity $K_{J_1J_2}$, often written as $(J_1||J_2||J)$, is called a reduced matrix element of the set of operator $T^{J,m}$ and is independent of the angular momentum projection numbers. Notice that the Clebsch-Gordan coefficient implicitly contains a $\Delta m_2$, which in turn guarantee conservation of angular momentum.
5. Gout formula.

This gives the integral of three Wigner rotation matrices as

\[
\int_0^{2\pi} \int_0^\pi \int_0^{2\pi} \frac{d\alpha}{\sin \beta} \sin \beta \frac{d\beta}{\sin \beta} \frac{d\gamma}{\sin \beta} \frac{1}{m_1 m_2 m_3} \frac{1}{m_4 m_5 m_6} \frac{1}{m_7 m_8 m_9} (\alpha \beta \gamma)^I D(\alpha \beta \gamma)^J D(\alpha \beta \gamma)^K =
\]

\[
= 8\pi^2 \frac{A_{m_1 m_2 m_3} A_{m_4 m_5 m_6} A_{m_7 m_8 m_9}}{m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8 m_9} \frac{C(\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8 \mu_9) \cdot C(\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8 \mu_9)}{(2\mu_1 + 1)}.
\]

(52)

4. TABLES OF CLEBSCH-GORDAN COEFFICIENTS FOR INTEGER ANGULAR MOMENTUM J \geq 0.5

Here we employ the notation

\[
C(\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8 \mu_9) = \begin{cases} R(N1/N2) \text{sgn}(M) \left(\frac{N1}{N2}\right)^{1/2} & \text{if } M = 0 \\ 0 & \text{otherwise} \end{cases}
\]

where \(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9\) are integers and the function \text{sgn}(M) gives the sign of its argument M. As mentioned before the results are also given for convenience in floating point form rounded to eight decimal places.
\[ D_{m_1}^{J_1}(\alpha \beta \gamma) D_{m_2}^{J_2}(\alpha \beta \gamma) = \sum_C C(J_1, J_2; m_1, m_2; m) C(J_1, J_2; n_1, n_2; n) D_{m_1}^{J_1}(\alpha \beta \gamma) D_{m_2}^{J_2}(\alpha \beta \gamma) \]  

(40)

In particular, since spherical harmonics \( Y_{Jm} \) are just special cases of Wigner rotation matrices

\[ D_{m_0}^{J}(\alpha \beta 0) = \left\{ 4\pi/(2J+1) \right\}^{1/2} Y_{Jm}(\alpha \beta \gamma) \]  

(41)

we have the useful coupling relation for spherical harmonics,

\[ Y_{J_1, m_1}(\alpha \beta \gamma) Y_{J_2, m_2}(\alpha \beta \gamma) = \sum_C \left\{ (2J_1+1)(2J_2+1)/4\pi(2J+1) \right\}^{1/2} \]  

\[ \times C(J_1, J_2; J; m_1, m_2, m) C(J_1, J_2; 0, 0, 0) Y_{Jm}(\alpha \beta \gamma) \]  

(42)

Remembering that \( D_{m_0}^{J}(00\theta) = P_J(\cos \theta) \) we also find at once the coupling relation for the Legendre polynomials \( P_j \), i.e.,

\[ P_{J_1}(\cos \beta) P_{J_2}(\cos \beta) = \sum_C C(J_1, J_2; J; 0, 0, 0) P_J(\cos \theta) \]  

(43)

Notice that the coupling of even rank polynomials only gives even rank \( P_J \) since (cf. eq. (37)) the Clebsch-Gordan coefficients \( C(J_1, J_2, J_3; 0, 0, 0) \) is zero unless \( J_1 + J_2 = J_3 \) is even.

Conversely we can decompose a Wigner rotation matrix as a linear combination of products of Wigner functions of lower rank,

\[ D_{m_0}^{J}(\alpha \beta \gamma) = \sum_C C(J_1, J_2; m_1, m_2, m) C(J_1, J_2; n_1, n_2, n) D_{m_1}^{J_1}(\alpha \beta \gamma) D_{m_2}^{J_2}(\alpha \beta \gamma) \]  

(44)

\[ \times \sum_C C(J_1, J_2; J, m_1, m_2, m) C(J_1, J_2; n_1, n_2, n) D_{m_1}^{J_1}(\alpha \beta \gamma) D_{m_2}^{J_2}(\alpha \beta \gamma) \]  

(45)

where the sum is extended to all indices not appearing on the left hand side.

3.6. Irreducible tensors coupling

An irreducible tensor operator of rank \( J \) can be defined as a set of \((2J+1)\) quantities \( T_{\alpha \beta \gamma}^J \) \((m = -J, -J+1, \ldots, J)\) which transform under the \((2J+1)\) dimensional representation of the full rotation group \( O^+(3) \) as
REFERENCES

(20) - M. Rotenberg, R. Bivins, N. Metropolis and J. K. Wooten Jr., The 3-j and 6-j Symbols (Technology Press MIT, 1959).
(21) - M. Abramovitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, 1965).
(22) - C. Bricman et al., Review of Particle Properties, Particle Data Group (CERN).
(23) - cf. for example CERN Library of computer programs and routines.
(29) - B. L. Van der Waerden, Die Gruppentheoretische Methode in der Quantenmecchanik (Springer, 1931).

(30) - B. L. Van der Waerden, Moderne Algebra (Springer, 1950).


(35) - L. C. Biedenharn, Tables of the Racah Coefficients, Oak Ridge National Laboratory, Physics Division, ORNL-1098 (1952).


(37) - C. Eckart, Rev. Mod. Phys. 2, 305 (1930).

(38) - V. A. Fox, ZETF (JETP) 10, 383 (1960)(in Russian).

(39) - A. Simon, Oak Ridge National Laboratory, ORNL-1718 (1954).


